QNo→ Suppose that T is a linear operator such that $T: X \to X$ where X is a finite dimensional normed linear space. Then all matrices representing T corresponding to various bases for X have the same eigenvalues.

Proof:- Let X be n-dimensional. Let $e = (e_1, e_2, \ldots, e_n)$ & $f = (f_1, f_2, \ldots, f_n)$ be two bases for X written as row matrices. Then every f_i can be written as a linear combination of $\{e_i\}$.

$\therefore$ Let $f_i = \sum_{j=1}^{n} \alpha_{ij} e_j, \quad i = 1, 2, \ldots n.$

Then $f = (f_1, f_2, \ldots, f_n) = \left(\sum_{j=1}^{n} \alpha_{1j} e_j, \sum_{j=1}^{n} \alpha_{2j} e_j, \ldots, \sum_{j=1}^{n} \alpha_{nj} e_j \right)$

$$= (e_1, e_2, \ldots, e_n) \begin{pmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & \cdots & \alpha_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{1n} & \alpha_{2n} & \cdots & \alpha_{nn} \end{pmatrix} = eC \quad \text{--- (1)}$$

Where, C is a non-singular n-rowed square matrix.

We can represent every $x \in X$ as a linear combination of elements with respect to the bases e & f.

$\therefore$ let $x = \sum_{j=1}^{m} \varepsilon_j e_j = e x_1$ & $x = \sum_{K=1}^{m} \eta_K f_K = f x_2$.

Where $x_1 = \{\varepsilon_1, \varepsilon_2, \text{------} \varepsilon_m\}$ & $x_2 = \{\eta_1, \eta_2, \text{----}, \eta_m\}$ are Column matrices.

$$\therefore x = e x_1 = \sum_{j=1}^{m} \varepsilon_j e_j = f x_2 = \sum_{K=1}^{m} \eta_K f_K \text{ —— } (2)$$

From (1) & (2), we have

$e x_1 = f x_2 = e C x_2$

$$\therefore x_1 = C x_2 \text{ —— } (3)$$

Similarly, if $Tx = y = e y_1 = f y_2$ then we shall have

$$y_1 = C y_2 \text{ —— } (4)$$

Now, $T(e_j)$, $j = 1, 2, \text{------}, m$ can be expressed as a linear combination of elements of e.

let, $T(e_j) = \sum_{i=1}^{m} \eta_{ij} e_i$, $j = 1, 2, \text{----} m$.

Therefore, from (2),

$$e y_1 = Tx = T\left(\sum_{j=1}^{m} \varepsilon_j e_j\right) = \sum_{j=1}^{m} \varepsilon_j T(e_j)$$

$$= \varepsilon_1 T(e_1) + \varepsilon_2 T(e_2) + \text{----} + \varepsilon_m T(e_m)$$

$$= \varepsilon_1 \sum_{i=1}^{m} \eta_{i1} e_i + \varepsilon_2 \sum_{i=1}^{m} \eta_{i2} e_i + \text{----} + \varepsilon_m \sum_{i=1}^{m} \eta_{im} e_i.$$

$$= e T_1 x_1.$$

Where, T_1 is the matrix which represent T with respect to the basis e.

Therefore, $y_1 = T_1 x_1$. Similarly, if T_2 denotes the matrix which represent T with respect to the basis f, then $y_2 = T_2 x_2$.

$$\therefore y_1 = T_1 x_1 \ \& \ y_2 = T_2 x_2 \quad \text{——— (5)}$$

From, (3), (4) & (5), we have

$$C T_2 x_2 = C y_2 = y_1 = T_1 x_1 = T_1 C x_2$$

Whence, operating by C^{-1}, we have

$$T_2 = C^{-1} T_1 C \quad \text{——— (6)}$$

With the help of this, we verify that the characteristic determinants of T_1 & T_2 are equal and from this the equality of eigen-values of T_1 & T_2. Using the fact det. (C^{-1}) det $C = 1$, we have

$$\det.(T_2 - \lambda I) = \det(C^{-1} T_1 C - \lambda C^{-1} I C), \text{ from (6)}$$

$$= \det(C^{-1}(T_1 - \lambda I)C)$$

$$= \det(C^{-1}) \det(T_1 - \lambda I) \det C.$$

$$= \det(T_1 - \lambda I).$$

This proves the theorem.